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Wave-induced boundary layers in a stratified fluid

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The boundary layers associated with gravity waves in a fluid with a linear variation of density are discussed in order to examine the steady Eulerian velocities induced by the Reynolds stress. For the case of a standing wave, the induced steady motion is shown to decay in an outer boundary layer which represents a balance between buoyancy and diffusion when the wave slope is sufficiently small but when viscous decay effects are even smaller. When the wave slope is larger, it would appear that two outer regions must be considered. Results for progressive waves are discussed only briefly, as they are qualitatively similar to the surface wave case.

1. Introduction

Thorpe (1968*a*, *b*) has discussed recently the modification of internal gravity waves due to second-order, finite amplitude effects. In order to supplement his discussion of the inviscid problem, we discuss here the modification of the flow due to viscous boundary layers existing at horizontal walls.

Rayleigh (1884) and Longuet-Higgins (1952) have demonstrated that mean Eulerian velocities can be generated in wave-induced motions due to such boundary layers, when terms of the second order in amplitude are considered. For the case of a progressive wave, the mean vorticity generated in the boundary layer diffuses throughout the main body of the fluid (cf. Longuet-Higgins 1960). For the case of a standing wave, as Rayleigh demonstrated in his discussion of the Kundt's dust tube phenomenon, the mean motion can decay, for a homogeneous fluid, in an outer region whose characteristic length is comparable to the wavelength. As discussed by Stuart (1966) and Riley (1965, 1967), Rayleigh's conclusion holds only when the Reynolds number associated with the steady induced flow is sufficiently small. When this condition does not hold, a second boundary layer can exist in which the mean motion decays. This layer, whose characteristic length can be much less than that of a wavelength but much greater than that of the inner Stokes layer, represents a balance between advection and diffusion and is therefore inherently non-linear in nature. For the case of a standing wave in a stratified fluid, we shall demonstrate that a second boundary layer can exist which represents a linear balance between buoyancy and diffusion.

2. Equations of motion and the inviscid solutions

The stratification is taken to arise from a linear variation of temperature between two parallel, horizontal walls, which bound the fluid and which are maintained at constant temperature. The Boussinesq approximation is made. The results should be representative of other smooth variations of density when this approximation is applicable.

Consider a co-ordinate system positioned on the lower wall, which is a distance H from the upper wall, and let $\tilde{x}$ and $\tilde{y}$ denote distance parallel and perpendicular to the wall, respectively. Let $\tilde{\epsilon}$ denote a characteristic disturbance amplitude, σ a characteristic frequency, $\tilde{\alpha}$ a wave-number, $\tilde{\psi}$ a stream function for our assumed two-dimensional motion, g the acceleration due to gravity which acts in the direction of negative $\tilde{y}$, and t time. We introduce the non-dimensional variables

$$\left.\begin{aligned} x = \tilde{\alpha}\tilde{x}, \quad y = n\pi\tilde{y}/H, \quad \tau = \sigma t, \\ \tilde{\psi} = (\tilde{\epsilon}\sigma H/n\pi)\,\psi, \quad \alpha = \tilde{\alpha}H/n\pi, \quad \epsilon = \tilde{\epsilon}\tilde{\alpha}, \end{aligned}\right\} \tag{2.1}$$

where

$$\tilde{u} = \partial\tilde{\psi}/\partial\tilde{y}, \quad \tilde{v} = -\partial\tilde{\psi}/\partial\tilde{x}. \tag{2.2}$$

The factor $(n\pi)$, n being an integer, is to be associated with the particular mode considered (the above definitions allow the increasing importance of viscosity with n to be seen explicitly in the equations).

The temperature is defined as

$$\tilde{T} = \tilde{T}_0[1 + (\gamma\tilde{y}/H) + (\epsilon\gamma/n\pi)\,T], \tag{2.3}$$

so that the density is given by the equation of state as

$$\tilde{\rho} = \tilde{\rho}_0[1 + \tilde{\beta}(\tilde{T} - \tilde{T}_0)] = \tilde{\rho}_0[1 + \beta\gamma\{(\tilde{y}/H) + (\epsilon/n\pi)\,T\}]. \tag{2.4}$$

The factors $\tilde{T}_0$ and $\tilde{\rho}_0$ are, respectively, a reference temperature and density, and we have defined

$$\beta = \tilde{\beta}\tilde{T}_0. \tag{2.5}$$

The pressure is defined as

$$\tilde{p} = \tilde{p}_0 - \tilde{\rho}_0 g[\tilde{y} + (\gamma\beta\tilde{y}^2/2H)] + (\tilde{\rho}_0\tilde{\epsilon}\sigma^2/\tilde{\alpha})\,p. \tag{2.6}$$

The momentum equations are

$$\frac{\partial^2\psi}{\partial\tau\,\partial y} + \epsilon\left\{\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\,\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\right\} = -\frac{\partial p}{\partial x} + \Lambda^{-1}\nabla^2\frac{\partial\psi}{\partial y} \tag{2.7}$$

and

$$\frac{\partial^2\psi}{\partial\tau\,\partial x} + \epsilon\left\{\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x^2} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y\,\partial x}\right\} = \alpha^{-2}\frac{\partial p}{\partial y} + \left(\frac{\gamma\beta g}{\sigma^2 H}\right)T + \Lambda^{-1}\nabla^2\frac{\partial\psi}{\partial x}, \tag{2.8}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial y^2} + \alpha^2\frac{\partial^2}{\partial x^2} \quad \text{and} \quad \Lambda^{-1} = \nu_0\,(n\pi)^2/\sigma H^2, \tag{2.9}$$

ν_0 being the kinematic viscosity, which, consistent with the Boussinesq approximation, is taken to be constant. The temperature equation is

$$\frac{\partial T}{\partial\tau} + \epsilon\left\{\frac{\partial\psi}{\partial y}\frac{\partial T}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial T}{\partial y}\right\} - \frac{\partial\psi}{\partial x} = (P\Lambda)^{-1}\nabla^2 T, \tag{2.10}$$

where P is the Prandtl number.

We assume that $\Lambda \gg 1$, $\epsilon \ll 1$, and expand as

$$\psi = \sum_{r,q} \epsilon^r \Lambda^{-\frac{1}{2}q} \psi_{rq}, \tag{2.11}$$

the expansion in half-powers of Λ being dictated by boundary-layer displacement effects. The governing equation for ψ_{00} is readily obtained as

$$\frac{\partial^2}{\partial \tau^2} \nabla^2 \psi_{00} - \left(\frac{\gamma\beta g \alpha^2}{\sigma^2 H}\right) \frac{\partial^2 \psi_{00}}{\partial x^2} = 0, \tag{2.12}$$

which admits solutions for standing waves, say, of the form

$$\psi_{00} = A \sin y \sin x \cos \tau, \tag{2.13}$$

$$T_{00} = A \sin y \cos x \sin \tau, \tag{2.14}$$

satisfying the inviscid wall boundary conditions and a condition of constant wall temperature if

$$\sigma^2 = -\frac{\gamma\beta g \alpha^2}{H(1+\alpha^2)}. \tag{2.15}$$

The frequency would have to be corrected for higher order viscous and non-linear effects, but our analysis will not extend to that order. Using (2·15), the vorticity equation can be expressed as

$$\left\{\frac{\partial}{\partial \tau} + \epsilon \left(\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\right)\right\} \nabla^2 \psi = -(1+\alpha^2)\frac{\partial T}{\partial x} + \Lambda^{-1}\nabla^4 \psi. \tag{2.16}$$

The equations for ψ_{10} and T_{10}, for the standing wave case, are

$$\frac{\partial}{\partial \tau} \nabla^2 \psi_{10} + (1+\alpha^2)\frac{\partial T_{10}}{\partial x} = 0 \tag{2.17}$$

and

$$\frac{\partial T_{10}}{\partial \tau} - \frac{\partial \psi_{10}}{\partial x} = \frac{A^2}{4} \sin 2y \sin 2\tau. \tag{2.18}$$

Particular solutions are

$$\psi_{10} \equiv 0$$

and

$$T_{10} = -\tfrac{1}{8}A^2 \sin 2y \cos 2\tau + \overline{T}_{10}(y). \tag{2.19}$$

We allow the existence of an arbitrary function $\overline{T}_{10}(y)$ because it cannot be determined further at this stage. In order to determine $\overline{T}_{10}(y)$, we would have to consider terms of $O(\epsilon\Lambda^{-1})$ in (2.10), at which order diffusive effects define the solution. In order to obtain the governing equation, the convective terms would require that the solution be determined through $O(\Lambda^{-\frac{1}{2}})$. But in order to obtain a solution at that stage, we would either have to allow for viscous decay of the wave (cf. Dore (1968)) or stipulate some forcing mechanism. We wish to avoid this complication and so suffer from our inability to define rigorously $\overline{T}_{10}(y)$. A condition which has been applied by Thorpe (1968*a*, p. 512; 1968*b*, pp. 582 and 588) results from stating that the unperturbed density of a particle originally at $\tilde{y}$ be equal to the density of the particle at $\tilde{y}+\tilde{\eta}$, where $\tilde{\eta}$ is the displacement, and, in order to close formally the problem, we choose to apply this (non-diffusive) condition. For the present case (cf. 2.3), we have

$$1 + (\gamma\tilde{y}/H) = 1 + (\gamma/H)(\tilde{y}+\tilde{\eta}) + \left(\frac{\epsilon\gamma}{n\pi}\right) T(\tilde{y}+\tilde{\eta}). \tag{2.20}$$

Expanding $\tilde{\eta}$ in terms of ϵ and using a Taylor's expansion to expand about the mean location, we find

$$\overline{T}_{10} = \tfrac{1}{8}A^2 \sin 2y, \tag{2.21}$$

in order to satisfy the condition.

For the case of a progressive wave described by

$$\left.\begin{aligned} \psi_{00} &= A \sin y \cos (x+\tau), \\ T_{00} &= A \sin y \cos (x+\tau), \end{aligned}\right\} \tag{2.22}$$

we find

$$T_{10} = \overline{T}_{10}(y) = \tfrac{1}{4}A^2 \sin 2y. \tag{2.23}$$

3. The Stokes layer solution for the stream function

We now consider the wall regions, where Stokes type boundary layers, with thickness of $O(\Lambda^{-\frac{1}{2}})$, are required. The form of the inviscid solutions suggest the definitions (cf. § 5 for further discussion of the temperature scaling)

$$\psi = \surd 2\Lambda^{-\frac{1}{2}}\Psi(X, x, \tau), \quad T = \surd 2\Lambda^{-\frac{1}{2}}\theta(Y, x, \tau), \tag{3.1}$$

where

$$Y = (\Lambda^{\frac{1}{2}}y/\surd 2). \tag{3.2}$$

Then (2.16) and (2.10) become

$$\left\{\frac{\partial}{\partial \tau} + \epsilon\left(\frac{\partial \Psi}{\partial Y}\frac{\partial}{\partial x} - \frac{\partial \Psi}{\partial x}\frac{\partial}{\partial Y}\right)\right\}\nabla_I^2\Psi = -2(1+\alpha^2)\Lambda^{-1}\frac{\partial \theta}{\partial x} + \tfrac{1}{2}\nabla_I^4\Psi \tag{3.3}$$

and

$$\frac{\partial \theta}{\partial \tau} + \epsilon\left\{\frac{\partial \Psi}{\partial Y}\frac{\partial \theta}{\partial \theta} - \frac{\partial \Psi}{\partial x}\frac{\partial \theta}{\partial Y}\right\} - \frac{\partial \Psi}{\partial x} = \frac{1}{2P}\nabla_I^2\theta, \tag{3.4}$$

where

$$\nabla_I^2 = \frac{\partial^2}{\partial Y^2} + 2\alpha^2\Lambda^{-1}\frac{\partial^2}{\partial x^2}. \tag{3.5}$$

Now ψ can be expanded in a manner similar to (2.11). We wish to emphasize the non-linear effects in the boundary layer and to avoid allowing for decay in the wave amplitude; we will therefore simply state that

$$\epsilon \gg \Lambda^{-\frac{1}{2}}, \tag{3.6}$$

so that the waves can be assumed to be of constant amplitude. On the basis of (3.3), it appears that buoyancy will have no effect upon the velocity field up to $O(\Lambda^{-1})$. Therefore, the steady flow induced by the Reynolds stress will be the same as for a homogeneous fluid, at least within the Stokes layer. As we shall show later, this statement is not necessarily valid outside the Stokes layer.

The pressure is constant up to $O(\Lambda^{-1})$ through the boundary layer so that it is simpler to work with the inner form of the x momentum equation (2.7). We then obtain for a standing wave

$$\frac{\partial^3 \Psi_{00}}{\partial Y^3} - 2\frac{\partial^2 \Psi_{00}}{\partial \tau \partial Y} = 2\frac{\partial p_{00}}{\partial x} = 2A \sin x \sin \tau \tag{3.7}$$

and

$$\frac{\partial^3 \Psi_{10}}{\partial Y^3} - 2\frac{\partial^2 \Psi_{10}}{\partial \tau \partial Y} = 2\frac{\partial p_{10}}{\partial x} + 2\left[\frac{\partial \Psi_{00}}{\partial Y}\frac{\partial^2 \Psi_{00}}{\partial Y \partial x} - \frac{\partial \Psi_{00}}{\partial x}\frac{\partial^2 \Psi_{00}}{\partial Y^2}\right], \tag{3.8}$$

where

$$\frac{\partial p_{10}}{\partial x} = -\frac{A^2}{4}\sin 2x\,(1+\cos 2\tau). \tag{3.9}$$

The well-known solution of Ψ_{00} is

$$\Psi_{00} = \tfrac{1}{2}A\sin x\{f_{00}(Y)\,e^{i\tau} + f_{00}^{*}(Y)\,e^{-i\tau}\}, \tag{3.10}$$

where the asterisk denotes the complex conjugate and

$$f_{00} = Y - \tfrac{1}{2}(1-i)\{1 - e^{-(1+i)Y}\}. \tag{3.11}$$

This function satisfies the no-slip wall boundary condition and matches the first term of the inner expansion of the outer solution for large Y.

The second-order problem is solved by taking

$$\Psi_{10} = \tfrac{1}{4}A^2 \sin 2x\,\{2\bar{\phi}_{10}(Y) + \phi_{10}(Y)\,e^{2i\tau} + \phi_{10}^{*}(Y)\,e^{-2i\tau}\} \tag{3.12}$$

and substituting into (3.8). The solutions are given, for example, by Stuart (1963, p. 384) as

$$\phi_{10} = \frac{1+i}{4\sqrt{2}}\,e^{-(1+i)Y\sqrt{2}} + \frac{iY}{2}\,e^{-(1+i)Y} - \frac{1+i}{4\sqrt{2}} \tag{3.13}$$

and

$$\bar{\phi}_{10} = C_1 Y^2 + \tfrac{13}{8} - \tfrac{3}{4}Y - \tfrac{1}{8}e^{-2Y} - \tfrac{3}{2}e^{-Y}\cos Y - e^{-Y}\sin Y - \tfrac{1}{2}Y e^{-Y}\sin Y. \tag{3.14}$$

We set $C_1 = 0$ on the basis that an acceptable match to a secondary boundary layer can thereby be made. For further discussion of this point, the reader is referred to Riley (1967, 1965).

For the case of a progressive wave (2.22), we let

$$\Psi_{00} = \tfrac{1}{2}A\{f_{00}(Y)\,e^{i(x+\tau)} + f_{00}^{*}(Y)\,e^{-i(x+\tau)}\}, \tag{3.15}$$

where $f_{00}(Y)$ is defined by (3.11). The equation for Ψ_{10} is given by (3.8) where

$$\frac{\partial p_{10}}{\partial x} = \tfrac{1}{2}A^2 \sin 2(x+\tau). \tag{3.16}$$

Let

$$\Psi_{10} = \tfrac{1}{2}A^2\{\bar{\phi}_{10}(Y) + i\phi_{10}e^{2i(x+\tau)} - i\phi_{10}^{*}e^{-2i(x+\tau)}\}. \tag{3.17}$$

Then $\phi_{10}(Y)$ is given by (3.13), while the steady velocity is (Phillips 1966, equation 3.4.25)

$$d\bar{\phi}_{10}/dY = -\tfrac{3}{2} + Ye^{-Y}(\cos Y + \sin Y) + e^{-Y}(2\cos Y - \sin Y) - \tfrac{1}{2}e^{-2Y}. \tag{3.18}$$

As both (3.14) and (3.18) indicate, the steady induced flows fail to decay within the Stokes layer. In the case of a standing wave, it will be demonstrated in the next section that the steady flow can be contained within a second boundary layer where a balance between buoyancy and diffusion exists. In contrast, no steady *vertical* component of velocity is generated by the progressive wave, and so buoyancy naturally cannot be directly influential in determining the streaming outside the Stokes layer. The outer flow problem would therefore be essentially similar to the flow induced near a wall by a free surface wave in a homogeneous fluid. Longuet-Higgins (1960) discussed how the vorticity generated in the boundary layer diffuses outward until a steady state is reached on a time scale $O(\Lambda)$ with respect to τ. This would exceed the time scale $O(\Lambda^{\frac{1}{2}})$ over which a free oscillation decays with time, and so some forcing mechanism must be stipulated in order to define a steady-state problem. The reader is referred to Kelly & Vreeman (1970) for an analysis of a problem when thermal forcing occurs at the wall; the following discussion will be mainly restricted to the case of a standing wave.

4. The outer viscous region for a standing wave

We now consider the behaviour of the steady, spatially periodic flow outside of the Stokes layer. The steady part of Ψ is of $O(\epsilon)$ outside the wall layer. With reference to the vorticity equation (2.16) we note that if stratification were absent the outer flow would be governed by the biharmonic equation if $\epsilon^2 \ll \Lambda^{-1}$. This situation corresponds essentially to that studied by Rayleigh (1884). The secondary flow would extend throughout a region of thickness $\tilde{\alpha}^{-1}$ or H, depending upon which length is smaller. For $\epsilon^2 \gg \Lambda^{-1}$, a secondary boundary layer, again for the homogeneous case, is possible whose thickness is $O(\epsilon^{-1}\Lambda^{-\frac{1}{2}})$, as demonstrated by Stuart (1966) and Riley (1965, 1967). In this region, a balance is made between advection and diffusion. When stratification is present, another possible balance arises, namely, that between buoyancy and diffusion. This balance appears to be possible in a region whose thickness is $O(\Lambda^{-\frac{1}{3}})$, i.e. somewhat greater than Stokes layer but typically much less than the width H. This layer is analogous to the $E^{\frac{1}{3}}$ (E being the Ekman number) Stewartson layer in homogeneous rotating fluids (cf. Veronis 1967).

We therefore introduce the variable

$$Z = \Lambda^{\frac{1}{3}} y \tag{4.1}$$

and define an appropriate stream function and temperature distribution in this region as

$$\psi = A \sin x \{\Lambda^{-\frac{1}{3}} Z \cos\tau - \Lambda^{-\frac{1}{2}} \cos(\tau - \tfrac{1}{4}\pi) + O(\Delta^{-1})\} + \epsilon\Lambda^{-\frac{1}{3}}\Psi^I(x, Z, \tau) \tag{4.2}$$

and
$$T = A \cos x \{\Lambda^{-\frac{1}{3}} Z \sin\tau + \Lambda^{-\frac{1}{2}} \cos(\tau + \tfrac{1}{4}\pi) + O(\Lambda^{-1})\} + \epsilon\theta^I(x, Z, \tau), \tag{4.3}$$

The terms in parentheses represent the lowest order term in the expansion of the inviscid solution together with the next order effect due to the Stokes layer (cf. (3.11) and (5.6)). As Z becomes small, we have from (3.14)

$$\Psi^I = -\tfrac{3}{8} A^2 Z \sin 2x. \tag{4.4}$$

Upon substitution of (4.2) and (4.3) into (2.16), cancelling terms which correspond to the linear inviscid solution, and applying (3.6) we can obtain

$$\Lambda^{\frac{1}{3}} \frac{\partial^2 \Psi^I}{\partial Z^2 \partial\tau} + \epsilon\Lambda^{\frac{1}{3}} \left\{ \left[A \sin x \cos\tau + \epsilon \frac{\partial \Psi^I}{\partial Z} \right] \frac{\partial}{\partial x} - A \left[\cos x \{ Z \cos\tau - \Lambda^{-\frac{1}{6}} \cos(\tau - \tfrac{1}{4}\pi)\} + \epsilon \frac{\partial \Psi^I}{\partial x} \right] \frac{\partial}{\partial Z} \right\} \frac{\partial^2 \Psi^I}{\partial Z^2}$$
$$= -(1+\alpha^2) \frac{\partial \theta^I}{\partial x} + \frac{\partial^4 \Psi^I}{\partial Z^4} + O(\Lambda^{-\frac{1}{3}}). \tag{4.5}$$

We expand Ψ^I and θ^I as

$$\Psi^I = \Psi^I_{00} + \epsilon\Psi^I_{10} + \Lambda^{-\frac{1}{3}}\Psi^I_{0,\frac{1}{3}} + O(\epsilon^2, \epsilon\Lambda^{-\frac{1}{6}}), \tag{4.6a}$$

$$\theta^I = \theta^I_{00} + \epsilon\theta^I_{10} + \Lambda^{-\frac{1}{3}}\theta^I_{0,\frac{1}{3}} + O(\epsilon^2, \epsilon\Lambda^{-\frac{1}{6}}). \tag{4.6b}$$

The equation for Ψ^I_{00} requires that Ψ^I_{00} be steady, as it is in view of the condition (4.4). It is clear that $O(1)$ terms in (4.5) will give an equation for Ψ^I_{00} and that this equation will consist of a linear balance between buoyancy and diffusion if $\epsilon^2\Lambda^{\frac{1}{3}} \ll 1$. The range of validity for the solution to be discussed is then

$$\Lambda^{-\frac{1}{2}} \ll \epsilon \ll \Lambda^{-\frac{1}{6}}. \tag{4.6c}$$

The equation which is obtained by taking the mean with respect to time of the $O(1)$ term is

$$\frac{\partial^4 \psi^I_{00}}{\partial Z^4} - (1+\alpha^2)\frac{\partial \theta^I_{00}}{\partial x} = 0. \tag{4.7}$$

The temperature equation can be expressed as

$$\Lambda^{\frac{1}{3}}\frac{\partial \theta^I}{\partial \tau} + \left\{\left[A\sin x\cos\tau + \epsilon\frac{\partial \Psi^I}{\partial Z}\right]\frac{\partial}{\partial x} - \left[A\cos x\left\{Z\cos\tau - \Lambda^{-\frac{1}{6}}\cos\left(\tau-\frac{\pi}{4}\right)\right\} + \epsilon\frac{\partial \Psi^I}{\partial Z}\right]\frac{\partial}{\partial Z}\right\}\{AZ\cos x\sin\tau + \epsilon\Lambda^{\frac{1}{3}}\theta^I\} - \frac{\partial \Psi^I}{\partial x} = \frac{1}{P}\frac{\partial^2 \theta^I}{\partial Z^2}. \tag{4.8}$$

The $O(\Lambda^{\frac{1}{3}})$ term in (4.8) requires that θ^I_{00} be steady. It will be demonstrated in the next section that the boundary condition at the outer edge of the thermal Stokes layer is such that this is true. If the mean of the $O(1)$ terms in (4.8) is taken, with the restriction (4.6c), we obtain

$$\frac{1}{P}\frac{\partial^2 \theta^I_{00}}{\partial Z^2} + \frac{\partial \Psi^I_{00}}{\partial x} = 0. \tag{4.9}$$

Combining (4.9) and (4.7), we obtain

$$\frac{\partial^6 \Psi^I_{00}}{\partial Z^6} + P(1+\alpha^2)\frac{\partial^2 \Psi^I_{00}}{\partial x^2} = 0. \tag{4.10}$$

If we let

$$\Psi^I_{00} = -\tfrac{1}{8}A^2\sin 2x\,\phi_s(Z), \tag{4.11}$$

then

$$\phi_s \sim e^{\lambda Z},$$

where

$$\lambda = \pm Q,\quad \left(\frac{-1\pm i\sqrt{3}}{2}\right)Q,\quad \left(\frac{1\pm i\sqrt{3}}{2}\right)Q \tag{4.12}$$

and

$$Q = \{4P(1+\alpha^2)\}^{\frac{1}{6}}. \tag{4.13}$$

A solution, composed of the terms which decay with Z and which satisfies the condition (4.4) as $Z \to 0$, is

$$\phi_s = \frac{2\sqrt{3}}{Q}e^{-\frac{1}{2}QZ}\sin\frac{\sqrt{3}}{2}QZ + K\left[e^{-QZ} - e^{-\frac{1}{2}QZ}\left\{\cos\frac{\sqrt{3}}{2}QZ - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}QZ\right\}\right], \tag{4.14}$$

where K is arbitrary at this stage. We note that ϕ_s decays to zero within the $\Lambda^{-\frac{1}{3}}$ layer. In comparison to Rayleigh's solution, it clearly indicates the strong inhibiting effect of gravity upon steady vertical motion in the stratified case. In comparison to the Riley–Stuart solution, it indicates the inhibiting effect of stratification upon separation as no solution of their type would be possible in homogeneous flow at those points at which advection of vorticity does not oppose diffusion.

If we now define

$$\theta_{00}^I = \frac{PA^2}{4} \cos 2x\, \theta_s(Z), \tag{4.15}$$

θ_s is determined from (4.9) as

$$\theta_s = -\frac{\sqrt{3}}{Q^3} e^{-\frac{1}{2}QZ}\left[\sin\frac{\sqrt{3}\,QZ}{2} - \sqrt{3}\cos\frac{\sqrt{2}\,QZ}{2}\right] + \frac{K}{Q^2}\left[e^{-QZ} + e^{-\frac{1}{2}QZ}\left\{\cos\frac{\sqrt{3}QZ}{2} + \frac{1}{2}\left(\sqrt{3} - \frac{1}{\sqrt{3}}\right)\sin\frac{\sqrt{3}\,QZ}{2}\right\}\right]. \tag{4.16}$$

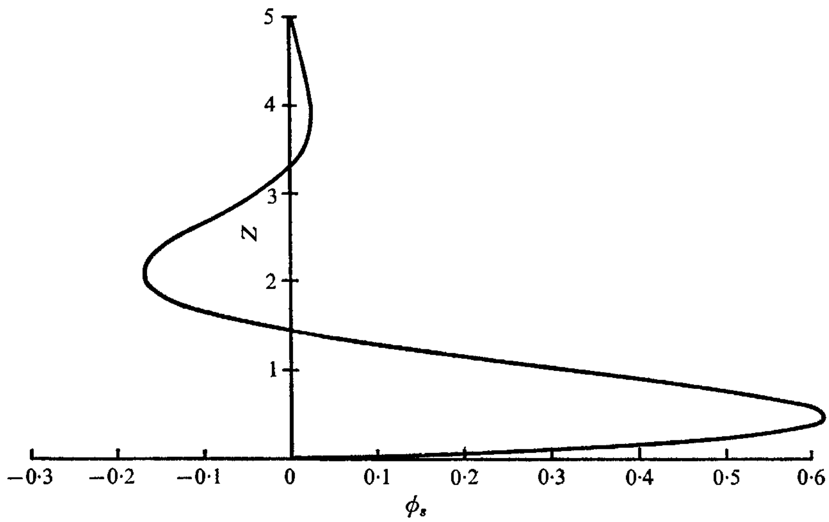

FIGURE 1. The function $\phi_s(Z)$, as defined by (4.9) for $Q = 2{\cdot}0$.

The constant K is determined by stipulating θ_{00}^I approaches zero as Z becomes small. If this were not true, we would have to solve the equation

$$d^2T/dY^2 = 0$$

within the Stokes layer, together with the boundary conditions that $T(0) = 0$ and that T approaches a constant for large Y. No such solution exists, and the resulting value of K is then found to be

$$K = -3/2Q. \tag{4.17}$$

With this result, $\phi_s(Z)$ and $\theta_s(Z)$ are shown in figures 1 and 2 for $Q = 2$. The buoyancy balance is clearly characterized by an oscillatory behaviour in the vertical direction of the induced mean motion.

When $\epsilon \gg \Lambda^{-\frac{1}{6}}$, a balance between advection and diffusion would at first seem to be appropriate for the outer region, which would then have the characteristic length $(\epsilon\Lambda^{\frac{1}{2}})^{-1}$. Buoyancy would exert only a higher order effect in this region, and so the governing equation for the mean flow in this region would be the same as for homogeneous flow, which has been discussed by Riley (1965). Riley obtained a solution to the resulting non-linear partial differential equation by expanding about a stagnation point of the steady flow where the flow is directed towards the wall. If his series solution is then used to solve for a temperature field, one finds that the temperature grows as (y/ϵ) at the outer edge of this region. Another region, with a characteristic length $\Lambda^{-\frac{1}{3}}$, is again required. Buoyancy, advection, and diffusion appear in the vorticity equation appropriate for this third layer, and the equation is similar to (4.5) when $(\epsilon^2\Lambda^{\frac{1}{3}})$ is of order unity. A numerical approach to the problem would appear to be required.

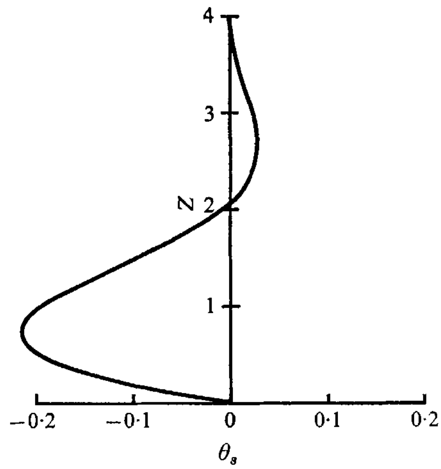

FIGURE 2. The function $\theta_s(Z)$ as defined by (4.14), for $Q = 2{\cdot}0$.

5. The Stokes layer solutions for the temperature distribution

We have been able to delay detailed discussion of the thermal Stokes layer up to this stage because it plays an essentially passive role, responding to velocity fluctuations in the wall layer and to temperature fluctuations in the inviscid and outer boundary-layer regions. The outer boundary layer plays a crucial role, however, in determining the expansion of the temperature for the case of standing waves (the reader is referred to a paper by Merkin (1967) in which a somewhat similar result was obtained when a non-linear balance was struck in the outer region for a flow induced by a cylinder with an unsteady temperature distribution). With reference to (3.4), we expand for a standing wave as

$$\theta(x, Y, \tau) = \theta_{00} + \epsilon\Lambda^{\frac{1}{3}}\,\theta_{1,\frac{1}{3}} + \epsilon\theta_{10} + \epsilon^2\Lambda^{\frac{1}{3}}\theta_{2,\frac{1}{3}} + O(\epsilon^3\Lambda^{\frac{1}{3}}), \tag{5.1}$$

where θ_{00} and $\theta_{1,\frac{1}{3}}$ must match, for large Y, to the leading terms of the inner expansions of (2.14) and (4.15), respectively. The lowest order modification of the steady temperature distribution within the Stokes layer is therefore controlled by the outer boundary layer. For $\Lambda^{-\frac{1}{3}} < \epsilon < \Lambda^{-\frac{1}{6}}$, this effect is more important than any other boundary-layer process, at least as far as the temperature is concerned. The vorticity equation remains unaffected by the buoyancy term up to $O(\epsilon\Lambda^{-\frac{2}{3}})$.

From (4.15)–(4.17) we have

$$\theta_{1,\frac{1}{3}}(x, Y) = -\frac{3YP}{8Q^2}\cos 2x \tag{5.2}$$

as Y becomes large. From (3.4), it is clear that (5.2) is the solution throughout the Stokes layer.

The functions θ_{00} and θ_{10} satisfy the equation

$$\frac{1}{P}\frac{\partial^2\theta_{00}}{\partial Y^2}-2\frac{\partial\theta_{00}}{\partial\tau}=-2\frac{\partial\Psi_{00}}{\partial x} \tag{5.3}$$

and

$$\frac{1}{P}\frac{\partial^2\theta_{10}}{\partial Y^2}-2\frac{\partial\theta_{10}}{\partial\tau}=-2\frac{\partial\Psi_{10}}{\partial x}+2\left\{\frac{\partial\Psi_{00}}{\partial Y}\frac{\partial\theta_{00}}{\partial x}-\frac{\partial\Psi_{00}}{\partial x}\frac{\partial\theta_{00}}{\partial Y}\right\}. \tag{5.4}$$

A solution of (5.3) which satisfies the condition $\theta(x,0,\tau)=0$ and matches the outer solution is

$$\theta_{00}=-\tfrac{1}{2}A\cos x\{g_{00}(Y)\,e^{i\tau}+g_{00}^*(Y)\,e^{-i\tau}\}, \tag{5.5}$$

where

$$g_{00}=iY-\left(\frac{1+i}{2}\right)-\left(\frac{1+i}{2}\right)\left(\frac{P}{1-P}\right)e^{-(1+i)Y}+\left(\frac{1+i}{2}\right)\left(\frac{1}{1-P}\right)e^{-(1+i)P^{\frac{1}{2}}Y}. \tag{5.6}$$

In order to solve (5.4), let

$$\theta_{10}=\tfrac{1}{4}A^2\bar{\bar{g}}_{10}(Y)+A^2\cos 2x\bar{g}_{10}(Y)-A^2\cos 2x\{g_{10}(Y)\,e^{2i\tau}+g_{10}^*(Y)\,e^{-2i\tau}\}$$
$$+\tfrac{1}{4}A^2\{\hat{g}_{10}(Y)\,e^{2i\tau}+\hat{g}_{10}(Y)\,e^{-2i\tau}\}, \tag{5.7}$$

where each of the functions must be zero at the wall. Upon substitution into (5.4), the solution for $\bar{\bar{g}}_{10}(Y)$ which matches (2.21) for large Y is found as

$$\bar{\bar{g}}_{10}(Y)=Y+\frac{1-P^{\frac{1}{2}}}{2(1-P^2)}+\frac{Pe^{-(1+P^{\frac{1}{2}})Y}}{2(1-P^2)}\begin{Bmatrix}(1-P^{\frac{1}{2}})\cos(1-P^{\frac{1}{2}})\,Y\\+(1+P^{\frac{1}{2}})\sin(1-P^{\frac{1}{2}})\,Y\end{Bmatrix}$$
$$+\left(\frac{P}{1-P}\right)Ye^{-Y}\cos Y-\left(\frac{P}{1-P}\right)e^{-Y}\sin Y$$
$$-\frac{P^{\frac{1}{2}}}{(1-P)}Ye^{-P^{\frac{1}{2}}Y}\cos P^{\frac{1}{2}}Y$$
$$+\frac{e^{-P^{\frac{1}{2}}Y}}{2(1-P)}\{(P^{\frac{1}{2}}-1)\cos P^{\frac{1}{2}}Y+(P^{\frac{1}{2}}+1)\sin P^{\frac{1}{2}}Y\}. \tag{5.8}$$

The function $(\bar{\bar{g}}_{10}(Y)-Y)$ is illustrated in figure 3 for two values of Prandtl number. This function represents the deviation within the wall region of the second-order mean-temperature distribution from the non-diffusive result. A maximum exists in this region, which suggests a mildly destabilizing influence.

The function $\hat{g}_{10}(Y)$ is matched for large Y to the unsteady part of T_{10} (2.19), whereas a solution for $g_{10}(Y)$ can be obtained so that it approaches a constant for large Y and consequently causes only a higher order effect on the outer flow; the solutions are

$$\hat{g}_{10}(Y)=C_2e^{-(1+i)(2P)^{\frac{1}{2}}Y}+\frac{(i-1)\,P(1+P^{\frac{1}{2}})}{4(1-P)\,(1+2P^{\frac{1}{2}}-P)}e^{-(1+i)(1+P^{\frac{1}{2}})\,Y}$$
$$+\frac{P^{\frac{1}{2}}}{2(1-P)}Ye^{-(1+i)\,P^{\frac{1}{2}}Y}+\frac{(i-1)\,(3+P^{\frac{1}{2}})}{4(1-P)}e^{(-1+i)\,P^{\frac{1}{2}}Y}$$
$$-\frac{(i-1)\,P^2}{4(2-P)\,(1-P)}e^{-2(i+i)Y}-\frac{P}{2(1-2P)}Ye^{-(1+i)Y}$$
$$+\frac{(1-i)P^2\,(4P-3)}{2(1-2P)^2\,(1-P)}e^{-(1+i)Y}-\tfrac{1}{2}Y+\frac{1-i}{4}, \tag{5.9}$$

where

$$C_2 = \frac{1-i}{4(1-P)}\left[\frac{P(1+P^{\frac{1}{2}})}{1+2P^{\frac{1}{2}}-P}+3+P^{\frac{1}{2}}+\frac{P^2}{P-2}-(1-P)-\frac{2P^2(4P-3)}{(1-2P)^2}\right], \quad (5.10)$$

and

$$g_{10}(Y) = C_3 e^{-(1+i)(2P)^{\frac{1}{2}}Y}+\frac{P}{8(1-P)}Ye^{-(1+i)Y}$$
$$+\frac{1-i}{8(1-P)(1-2P)}e^{-(1+i)Y}-\frac{P^{\frac{1}{2}}}{8(1-P)}Ye^{-(1+i)P^{\frac{1}{2}}Y}$$
$$+\frac{(1-i)(1+P^{\frac{1}{2}})}{16(1-P)}e^{-(1+i)P^{\frac{1}{2}}Y}+\frac{(1-i)P}{16\sqrt{2}(1-P)}e^{-(1+i)Y\sqrt{2}}$$
$$+\frac{(1-i)P(1-P^{\frac{1}{2}})}{16(1-P)(1+2P^{\frac{1}{2}}-P)}e^{-(1+i)(1+P^{\frac{1}{2}})Y}+\frac{1-i}{16\sqrt{2}}, \quad (5.11)$$

where

$$C_3 = \frac{i-1}{16(1-P)}\left[\frac{2}{1-2P}+1+\frac{\sqrt{2}}{2}+P^{\frac{1}{2}}+\frac{P(1-P^{\frac{1}{2}})}{(1+2P^{\frac{1}{2}}-P)}\right]. \quad (5.12)$$

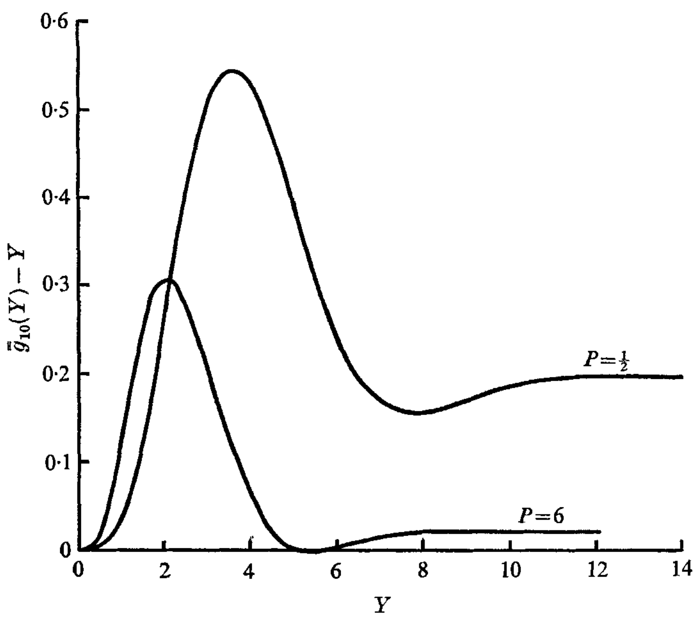

FIGURE 3. The behaviour of the second-order correction to the mean temperature distribution within the boundary layer.

The function $\bar{g}_{10}(Y)$ is of special interest, because it is forced in part by the steady velocity field. In fact, the steady velocity field gives rise to the most rapidly growing part of $\bar{g}_{10}(Y)$. The solution is

$$\bar{g}_{10}(Y) = \frac{P}{4}Y^3-\frac{11P}{8}Y^2+C_4Y+C_5+\frac{P}{16}\left(\frac{1-3P}{1-P}\right)e^{-2Y}+\frac{P}{4}\left(\frac{1-2P}{1-P}\right)Ye^{-Y}\cos Y$$
$$+\frac{P}{4}\left(\frac{5-4P}{1-P}\right)e^{-Y}\cos Y-\frac{P}{2}\left(\frac{3-2P}{1-P}\right)e^{-Y}\sin Y-\frac{P^{\frac{1}{2}}}{4(1-P)}Ye^{-P^{\frac{1}{2}}Y}\cos P^{\frac{1}{2}}Y$$

$$+\frac{(P^{\frac{1}{2}}-3)}{8(1-P)}e^{-P^{\frac{1}{2}}Y}\cos P^{\frac{1}{2}}Y-\frac{P[1-3P^{\frac{1}{2}}-3P+P^{\frac{3}{2}}]}{8(1-P)(1+P)^2}e^{-(1+P^{\frac{1}{2}})Y}\cos(1-P^{\frac{1}{2}})Y$$
$$+\frac{P^{\frac{1}{2}}+3}{8(1-P)}e^{-P^{\frac{1}{2}}Y}\sin P^{\frac{1}{2}}Y-\frac{P[1+3P^{\frac{1}{2}}-3P-P^{\frac{3}{2}}]}{8(1-P)(1+P)^2}e^{-(1+P^{\frac{1}{2}})Y}\sin(1-P^{\frac{1}{2}})Y, \tag{5.13}$$

where C_5 is determined so that $\bar{g}_{10}(0) = 0$ but where C_4 can be determined only through the consideration of higher order effects in the outer boundary layer. The leading term in (5.13) matches to the $O(Z^3)$ term in the inner expansion of (4.15) (the $O(Z^2)$ term in that expansion is zero), but the term of $O(Y)$ in (5.13) would force us to revise the expansion (4.6*a, b*) so as to include an $O(\Lambda^{-\frac{1}{3}})$ term which does not depend on time. This modification, however, does not affect the determination of the lowest order solution in the outer boundary layer.

The $O(\epsilon^2\Lambda^{\frac{1}{3}})$ term in (5.1) arises from the non-linear interaction of Ψ_{00} and $\theta_{1,\frac{1}{3}}$ and therefore fluctuates with the fundamental frequency. One can show that it grows as Y at the edge of the Stokes layer and therefore presumably matches to the $O(\epsilon)$ term in the expansion (4.6*b*) and decays within the outer boundary layer.

For the case of a progressive wave, where the streamfunction is given by (3.15), (3.17), we expand the temperature as in (5.1) with $\theta_{1,\frac{1}{3}} = \theta_{2,\frac{1}{3}} \equiv 0$. If we let

$$\theta_{00} = -\tfrac{1}{2}iA\{g_{00}(Y)\,e^{i(x+\tau)}+g^*_{00}(Y)\,e^{-(x+\tau)}\}, \tag{5.14}$$

then $g_{00}(Y)$ is given by (5.6). Further, if we let

$$\theta_{10} = \tfrac{1}{2}A^2\bar{\bar{g}}_{10}(Y)+2A^2g_{10}(Y)\,e^{2i(x+\tau)}+2A^2g^*_{10}(Y)\,e^{-2i(x+\tau)}, \tag{5.15}$$

then $\bar{\bar{g}}_{10}(Y)$ is given by (5.8), allowing a match to (2.23), and $g_{10}(Y)$ is given by (5.11).

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